Duality II

<u>Today</u> $\mathrm{Pic}^\circ_{X/k}$ is representable.

§1 Lifting group actions

<u>Def</u> S, X/S schemes, G/S group scheme

+ action $\mu : G \times_S X \to X$.

$\mathcal{F}$ qcoh $\mathcal{O}_X$-module.

<u>Lifting of G-action to $\mathcal{F}$</u> $\overset{=}{\text{def}}$

$$\phi : \mu^* \mathcal{F} \xrightarrow{\simeq} p^* \mathcal{F} \qquad p = p_X$$

<u>s.th.</u>

$$\begin{tikzcd}
(\mathrm{id},\mu)^* \mu^* \mathcal{F} \arrow[rr, "(\mathrm{id},\mu)^*\phi"] \arrow[dr, "(m,\mathrm{id})^*\phi"'] & & p_{23}^* \mu^* \mathcal{F} \arrow[dl, "p_{23}^* \mathcal{F}"] \\
& p_3^* \mathcal{F} &
\end{tikzcd}$$

@

Explanation T/S, $g \in G(T)$

$$\begin{array}{ccc} (g, id_X)^* \mu^* \mathcal{F} & \xrightarrow{(g, id_X)^* \phi} & (g, id_X)^* p^* \mathcal{F} \\ \| & & \| \\ g^* \mathcal{F}_T & \xrightarrow[\cong]{\phi_g} & \mathcal{F}_T \end{array}$$

Here: $\mathcal{F}_T = p_X^* \mathcal{F}$ on X_T

$$\begin{tikzcd} g: X_T \arrow[rr] \arrow[dr] & & X_T \arrow[dl] \\ & T & \end{tikzcd}$$

ⓐ translates to $\phi_{hg} = \phi_h \circ h^* \phi_g$ ⓐ

which means that "G acts on geometric space of $\mathcal{F}$".

E.g. $G(T)$ now acts on $\mathcal{F}(U)$ whenever $G(T) \cdot U = U$.

$$
\begin{tikzcd}
\mathcal{F}(U) \arrow[r, "g^*"] \arrow[drr, bend right] & g^*\mathcal{F}(g^{-1}U) \arrow[r, phantom, "\overset{\text{assumption on } U}{=}"] & g^*\mathcal{F}(U) \arrow[d, "\phi_g"] \\
& & \mathcal{F}(U)
\end{tikzcd}
$$

@ means that this is a group action.

<u>Rmk</u> Given $\{\phi_g\}_{T/S,\ g \in G(T)}$

satisfying @ + satisfying $\phi_{g \circ u} = u^*\phi_g$

$\forall\, u : T' \to T$,

can recover $\phi : \mu^*\mathcal{F} \xrightarrow{\cong} p^*\mathcal{F}$

from case $T = G$, $g = \mathrm{id}_G$

Cocycle cond @ for ϕ follows from case $T = G \times_S G$ + use of functoriality of $\{\phi_g\}$ + @

§ 2 Descent G/S finite loc free

μ free (i.e. $G \times_S X \longrightarrow X \times_S X$
$(g, x) \longmapsto (gx, x)$

is a closed immersion)

$X \xrightarrow{\pi} Y := X/G.$ Recall

i) $X \times_Y X \xleftarrow{\cong} G \times_S X$

ii) π finite & loc free,

in particular fpqc.

Last time fpqc descent datum

= lifting of G-action.

Cor (of fpqc descent) Equivalence

$\mathrm{QCoh}_Y \xrightarrow{\cong} \{G$ qcoh $\mathcal{O}_X$-mod

$+$ lifting $\phi : \mu^* \mathcal{F} \to p^* \mathcal{F} \}$

$$\mathcal{E} \longmapsto \pi^* \mathcal{E} + \text{natural } \phi$$

$$(\pi_* \mathcal{F})^G \longleftarrow \mathcal{F}$$

i.e. $U \longmapsto \ker\left(\mathcal{F}(\pi^{-1}U) \xrightarrow{\phi \circ \mu^* - p^*} p^*\mathcal{F}(G \times_S \pi^{-1}U)\right)$

<u>Rule</u> $\mathcal{F} + \phi : p_1^* \mathcal{F} \xrightarrow{\cong} p_2^* \mathcal{F}$

for general fpqc : $X \xrightarrow{\pi} Y$

Descended module is

$$\mathcal{E}(U) = \ker\left(\mathcal{F}(\pi^{-1}U) \xrightarrow{\phi \circ p_1^* - p_2^*} (p_2^*\mathcal{F})(\pi^{-1}(U \times \pi^{-1}U))\right)$$

<u>Examples</u>

1) $\mu^* \mathcal{O}_X \xrightarrow[\cong]{\mu^*} \mathcal{O}_{G \times_S X} \overset{\phi = \mathrm{id}}{=} \mathcal{O}_{G \times_S X}$

$$\uparrow \cong$$
$$p^* \mathcal{O}_{G \times_S X}.$$

2) T/S, $g \in G(T)$

$$\phi_g : g^* \Omega^1_{X_T/T} \xrightarrow{\cong} \Omega^1_{X_T/T}$$
$$g^* df \longmapsto d(g^* f)$$

Provide a 2^{nd} proof of fact that

$$\Omega^1_{G/S} \cong p^* (p_* \Omega^1_{G/S})^G$$

$\Rightarrow$ pull-back from $\mathcal{O}_S$-module.

§3 <u>Poincaré bundle</u> $k = \bar{k}$

X/k AV, $\mathcal{L}$ ample on X

$X^\vee := X/K_{\mathcal{L}}$

$$X \times X \longrightarrow X^\vee \times X \quad \text{quotient by } K_{\mathcal{L}} \times 0$$

$$\mathcal{M} := m^*\mathcal{L} \otimes p_1^*\mathcal{L}^{-1} \otimes p_2^*\mathcal{L}^{-1}$$

on $X \times X$.

Satisfies

$$\mathcal{M}|_{\{x\}\times X} = \mathcal{M}|_{X \times \{x\}} \cong t_x^*\mathcal{L} \otimes \mathcal{L}^{-1}$$

Defining property of $K_{\mathcal{L}}$ is

$$\mu^*\mathcal{L} \simeq p_X^*\mathcal{L} \qquad \mu, p : K_{\mathcal{L}} \times X \to X$$

$$\Longrightarrow \exists\, \phi : (\mu, \mathrm{id}_X)^*\mathcal{M} \simeq p_{X\times X}^*\mathcal{M}$$

on $K_{\mathcal{L}} \times X \times X$

Choices of such ϕ are torsor under

$$H^0(K_{\mathcal{L}} \times X \times X, \mathcal{O})^\times = H^0(K_{\mathcal{L}}, \mathcal{O})^\times.$$

<u>Claim</u> ϕ may be chosen to satisfy @.

Fix $\mathcal{M}|_{X\times\{0\}} \underset{\theta}{\cong} \mathcal{O}_X$

Then, $\exists!\ \phi$ s.th.

$$\phi|_{K_{\mathcal{L}}\times X\times\{0\}} : \begin{tikzcd} \mu^*(\mathcal{M}|_{X\times\{0\}}) \arrow[rr, "\cong"] \arrow[d, "\theta"', "\cong"] & & p^*(\mathcal{M}|_{X\times\{0\}}) \arrow[d, "\theta", "\cong"'] \\ \mu^*\mathcal{O}_X \arrow[rr, equal] & & p^*\mathcal{O}_X \end{tikzcd}$$

because

$$H^0(K_{\mathcal{L}}\times X\times X, \mathcal{O})^\times \xrightarrow{\cong} H^0(K_{\mathcal{L}}\times X\times\{0\}, \mathcal{O})^\times$$

Then cocycle cond is satisfied.

Def $\mathcal{P}$ on $X^\vee\times X$ defined as descent of $(\mathcal{M}, \phi)$.

§4 The Theorem

<u>Thm</u> $(X^\vee, P)$ represents $\mathrm{Pic}^\circ_{X/k}$, i.e.

$\forall\, S/k \quad \forall\, Q \in \mathrm{Pic}^\circ_{X/k}(S)$ s.th.

$$Q\big|_{O_S \subseteq X_S} \cong \mathcal{O}_S$$

$\exists!$ morphism $u_Q : S \longrightarrow X^\vee$

s.th. $Q = (u_Q, \mathrm{id}_X)^* P$.

<u>Proof</u> $S/\operatorname{Spec} k$, $Q \in \mathrm{Pic}^\circ_{X/k}(S)$

s.th. $Q|_{O_S} \cong \mathcal{O}_S$.

On $X^\vee_S \times_S X_S = S \times X^\vee \times X$

consider $\mathcal{M} := p_{13}^* Q^{-1} \otimes p_{23}^* P$

$\Gamma \subseteq S \times X^\vee$ closed subscheme s.th.

$T \xrightarrow{v} S \times X^\vee$ factors through Γ

$\iff v^* \mathcal{M} \cong p_T^* \mathcal{D}$ for $\mathcal{D} \in \mathrm{Pic}(T)$ on $T \times X$

$\iff v^* \mathcal{M} \cong \mathcal{O}_{T \times X}$ (because $\mathcal{M}|_{S \times X \times \{0\}} \cong \mathcal{O}_{S \times X}$.)

<u>To show</u>

<u>Note</u> $(s, y) \in \Gamma$
$\iff Q(s) \cong P(y)$.

Γ graph Γ_u for some $u : S \to X^\vee$

$\iff \Gamma \xrightarrow{pr} S$ is an isomorphism.

<u>Thm</u> from last time:

$\phi_{\mathcal{L}}(k) : X(k) \longrightarrow \mathrm{Pic}^\circ(k)$

induces $X^\vee(k) \cong \mathrm{Pic}^0(k)$

$\Longrightarrow$ (allowing k to be any alg closed field) $|\Gamma| \longrightarrow |S|$ is a bijection.

$(s, u(s)) \longmapsto s$.

<u>Rmk</u> If char $k = 0$ & S normal variety, then already sufficient. (Zariski's Main Thm)

<u>Step 1</u> Reduce to S artinian k-alg.

$\Gamma \to S$ being iso is local on S, so wlog S affine.

$\Gamma \to S$ q-finite + proper $\Longrightarrow$ affine

$\Longrightarrow \Gamma$ also affine.

Enough: $\mathcal{O}_{S,s} \xrightarrow{\cong} \mathcal{O}_{\Gamma,(s,u(s))}$

Formation of Γ commutes w/ bc along $k \to \overline{\kappa(s)}$.

$\Rightarrow$ wlog $s \in S(k)$.

Then $\mathcal{O}_{S,s} \xrightarrow{\cong} \mathcal{O}_{\Gamma,(s,x(s))}$ iso

$\Leftrightarrow$ iso mod $\mathfrak{m}_s^n$ $\forall n$

(Standing assumption: Schemes are loc noetherian)

So wlog, $S = \operatorname{Spec} A$, A artinian local k-algebra.

Further reduction $x(s) = 0 \in X^\vee(k)$

by replacing Q by $Q \otimes Q(s)^{-1}$.

to assume $Q(s) = \mathcal{O}_X$

<u>Step 2</u> $H^i(S \times X^\vee \times X, \mathcal{M}) = 0$
if $i \neq g$.

Consider all $R^i p_{12,*} \mathcal{M}$. Have support at $(s,0)$ since $\mathcal{M}|_{s \times y \times X} \cong \mathcal{O}_X$

$\iff y = 0$ in $X^\vee(k)$.
by defn of $X^\vee$

+ Lem. from last time that
$H^i(X, \mathcal{D}) = 0 \quad \forall i$ if
$\mathcal{D} \in \mathrm{Pic}^0(X), \mathcal{D} \not\cong \mathcal{O}_X$.

In partic, $R^i p_{12,*} \mathcal{M}$ supported on an affine, so $H^j(R^i p_{12,*} \mathcal{M}) = 0$
if $j > 0$.

$\xrightarrow{\text{Leray}}$ $H^i(S \times X^\vee \times X, \mathcal{M})$

$$= H^0(S \times X^\vee, R^i p_{12,*}\mathcal{M})$$

$\leadsto$ finite dim k-vsp + $\neq 0$ in degree at most $0 \leq i \leq g$.

Let $0 \to K^0 \to \cdots \to K^g \to 0$

perfect complex of $A \otimes_k \mathcal{O}_{X^\vee,0}$-modules

computing $R^i p_{12,*}\mathcal{M}$ universally.

The following Lemma applies:

<u>Lem</u> $\mathcal{O}$ reg loc ring of dim g.

$0 \to K^0 \to \cdots \to K^g \to 0$ perfect $\mathcal{O}$-complex

s.th. $H^i(K^\bullet)$ artinian $\forall i$.

Then $H^i(K^\bullet) = 0$ for $i < g$.

Proof $x \in \mathfrak{m} \setminus \mathfrak{m}^2$ ($\mathfrak{m} \subseteq \mathcal{O}$ max ideal)

Then $\mathcal{O}/x\mathcal{O}$ regular of dim $g-1$.

$$0 \longrightarrow K^\bullet \xrightarrow{x} K^\bullet \longrightarrow \overline{K}^\bullet \longrightarrow 0$$

gives long seq in cohom:

$$H^i(K^\bullet) \xrightarrow{x} H^i(K^\bullet) \longrightarrow H^i(\overline{K}^\bullet) \longrightarrow$$

$$H^{i+1}(K^\bullet) \xrightarrow{x} H^{i+1}(K^\bullet)$$

$\Longrightarrow$ $H^i(\overline{K}^\bullet)$ artinian again

Induction

$\Longrightarrow$ $H^i(\overline{K}^\bullet) = 0 \quad \forall\, i < g-1.$

$\Longrightarrow$ $x \cdot : H^i(K^\bullet) \hookrightarrow H^i(K^\bullet)$

$\forall\, i < g.$

Since $H^i(K^\bullet)$ artinian, implies

$H^i(K^\bullet) = 0$. $\square$

Lemma + Step 2.

Step 3 $\quad H^i(S \times X^\vee \times X, \mathcal{M})$ free

A-module

Same as with $R^i p_{12,*} \mathcal{M}$,

$R^i p_{13,*} \mathcal{M}$ are supported on the finite

set $|S \times K_{\mathcal{L}}|$, so

$$H^i(S \times X^\vee \times X, \mathcal{M}) = H^0(S \times X, R^i p_{13,*} \mathcal{M})$$

Now

$$R^i p_{13,*} \mathcal{M} \overset{\text{Def of } \mathcal{M}}{=} R^i p_{13,*}(p_{23}^* P \otimes p_{13}^* Q^{-1})$$

$$= R^i p_{13,*}(p_{23}^* P) \otimes Q^{-1}$$

by projection formula.

non-canonically

$$\cong R^i p_{13,*}(p_{23}^* P).$$

since $\mathcal{Q}|_{\text{finite set}} \cong \mathcal{O}_{\text{finite set}}$.

$$\Longrightarrow H^i(S \times X^\vee \times X, \mathcal{M})$$

$$\simeq A \otimes_k H^i(X^\vee \times X, P).$$

is free as an A-module. $\square$

Step 3

<u>Preparation for Finale</u> (back to $R^i p_{12,*}$)

$$\mathcal{B} = A \otimes_k \mathcal{O}_{X^\vee, 0} = \mathcal{O}_{S \times X^\vee, (s,0)}.$$

$$0 \to K^0 \to \cdots \to K^g \to 0$$

perfect complex of $\mathcal{B}$-modules computing $R^i p_{12,*} \mathcal{M}$ universally.

Then $\ker\left(K^0((s,0)) \longrightarrow K^1((s,0))\right)$

$$= H^0(\mathcal{M}|_{s \times 0 \times X}) \cong k \quad *$$

$$\hat{K}^i := \mathrm{Hom}_B(K^i, B) \qquad \text{cokernel}$$

$$\hat{K}^1 \longrightarrow \hat{K}^0 \longrightarrow C \longrightarrow 0$$

Then $C((s,0)) \cong \mathrm{Hom}_B(k,k)$

has k-dim 1.

$\overset{\text{Nakayama}}{\Longrightarrow}$ C generated by single elt as B-module.

$\Longleftrightarrow$ $C \cong B/\mathfrak{b}$.

<u>Rmk</u> Compare this with discussion in Lect 19:

Get $\Gamma = \mathrm{Spec}\, B/\mathfrak{b}$.

<u>Finale</u>

$$\begin{tikzcd} A \arrow[r] \arrow[dr, "\cong"'] & B \arrow[d] \\ & B/\mathfrak{b} \end{tikzcd}$$

Composition is Iso, i.e. $S \cong \Gamma$.

Injectivity

$$0 \to \hat{K}^g \to \cdots \to \hat{K}^0 \to 0$$

has artinian cohom like $K^\bullet$

Lemma from Step 2 $\Longrightarrow$

$$H^i(\hat{K}^\bullet) = \begin{cases} 0 & i \neq g \\ C & i = g \end{cases}$$

<u>Claim</u> C $\mathfrak{b}$-torsion implies that cohomology of $K^\bullet = \hat{\hat{K}}^\bullet$ is also $\mathfrak{b}$-torsion.

<u>Proof</u> Show that mult. by $b \in \mathfrak{b}$ is null-homotopic. This is preserved under dualizing.

$$\begin{tikzcd}
\hat{K}^1 \arrow[r, "d"] \arrow[d, "b = b^1"'] & \hat{K}^0 \arrow[r] \arrow[d, "b^0 = b"] \arrow[dl, "\phi"'] & 0 \\
\hat{K}^1 \arrow[r, "d"] & \hat{K}^0 \arrow[r] & 0
\end{tikzcd}$$

$b \, C = 0 \iff \operatorname{Im}(b^0) \subseteq \operatorname{Im} d$

$\hat{K}^0$ projective $\implies \exists \, \phi$ as in diagram

This is iterable to construct homotopy. $\square$ Claim.

<u>Step 2+3</u> Only column of $K^\bullet$ is K^g/K^{g-1}

which is $\neq 0$ & A-free.

Claim

$\implies$ $A \cap b = 0 \iff A \hookrightarrow B/b$.

$\square$ Injectivity

<u>Surjectivity</u> By Nakayama, enough

to see $A/m_A \twoheadrightarrow B/b + m_A B$.

$\Longrightarrow$ wlog $S = \operatorname{Spec} \kappa(s)$ $(= \operatorname{Spec} k)$

Then we ask for maximal $\Gamma \subseteq X^\vee$

over which P trivial.

This is $\{0\} = K_{\mathcal{L}} / K_{\mathcal{L}}$ by construction.

Thm.

<u>Used in Step 1</u>

$Y \longrightarrow S$ flat + proper + geom int fibers.

$\mathcal{L}$ on Y.

$\xrightarrow{\S 19}$ $Z = Z(\mathcal{L}) \subseteq S$ s.th

$T \longrightarrow S$ factors through Z

$\iff$ $\mathcal{L}_T \in \mathrm{Pic}(Y_T)$

lies in $p_T^* \mathrm{Pic}(T)$.

Then $\forall S' \longrightarrow S$,

$$Z(\mathcal{L}_{S'}) = S' \times_S Z.$$